Political Economy and Social Welfare with Voting Procedure

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ABSTRACT

Mathematical Economics, Social Science and Political Science are inter-related. In this paper, an attempt has been made to describe aspects of these subjects by introducing examples, definitions, mathematical calculations and discussions. Game Theory is included in this paper to study mathematical models in economics and political science especially to study Nash equilibrium. Success and failure of democracy are interpreted as different equilibria of a dynamic political game with cost of changing leadership. Unitary democracy can be frustrated when voters do not replace corrupt leaders. Federal democracy cannot be consistently frustrated at both national and provincial levels. Arrow's theorem indicates that the aggregate of individuals' preferences will not satisfy transitivity, indifference to irrelevant alternatives and non-dictatorship, simultaneously to enable one of the individuals becomes a dictator. In this paper both social welfare functions and social choice correspondence are considered in economical environments.

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1. INTRODUCTION

This paper is about social welfare function, which is introduced by Arrow's book (1951, 1963). It has provided striking answer to a book abstract problem of democracy. A social welfare function is a procedure for aggregating properties of individual preferences into social orderings. Arrow's theorem shows that it is impossible for a social welfare function to satisfy five conditions namely: i) Completeness and Transitivity, ii) Universality, iii) Pareto Consistency, iv) Independence of Irrelevant Alternatives and v) Non-dictatorship simultaneously. Arrow's theorem makes social choice theory more challenging and interesting, as it makes apparent the inadequacy of specific approach to construct reasonable social welfare functions from simpler aggregation rules which we call social choice functions, so that one can use the solution to a certain aggregation problem to solve more complex aggregation problems. We have tried to describe social problems in simple way. In the start a simple story on game theory (Fudenberg and Tirole 1991) and then following

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(Myerson 1996 and 2004) game theoretical model named 'Battle of Sexes' used to describe how the political institutions are formed and how these are developed to create efficient political leaders. The paper includes aspects of both federal and unitary democracy following Myerson (2006, 2009) where we have shown that both democracies have some difficulties but comparatively federal democracy is better. Although both democracies are not stainless, the citizens accept them. Peoples franchise for their most preferred person and they can oust or not re-elect a corrupt leader again. A brief description is given on the behavior between the two adversary countries following Myerson (2006, 2008 and 2009). To describe this we have proceeded with Prisoner dilemma game. Here we have showed that a powerful country say America (A) cannot attack a comparatively weaker country. If A attacks, it has to lose some reputation which is bad for A. The whole world will see A's behavior carefully and final result will either lose its reputation or become popular depending on A's bad or good behavior.

Some definitions are included from Arrow (1963); Sen (1970); Islam (1997 and 2008); Islam, Mohajan and Moolio (2009: Spring). These will be helpful for those who are new in this field. A brief discussion on Arrow's impossibility theorem is given to understand the full concept of theorem following Reny (2000); Geanokoplos (2005); Islam, Mohajan and Moolio (2009: Spring). Arrow's theorem plays an important role in society so that from this paper even a layoff can realize the importance of the theorem. Nash equilibrium is an essential part in Game Theory. We tried to give a simple description of Nash equilibrium following Nash (1951); Fudenberg and Tirole (1991); Myerson (1985, 1996, 2004, 2006 and 2009).

Some other related studies are: Black (1948 and 1958); Arrow (1951 and 1963); Schelling (1960); Barbera (1980 and 2001); Barbera and Coelho (2009); Barbera, Berga and Moreno (2009); Barbera and Moreno (2008); Feldman (1974); Harsanyi (1973); Hardin (1989); Sen (1970); Myerson (1996, 2007 and 2009); Blackorby, Donaldson and Weymark (1990); Reny (2000); Bossert and Weymark (2003); Geanokoplos (2005); Bretyon and Weymark (2006); Feldman and Serrano (2006, 2007 and 2008), Islam (1997 and 2008); Ehlers and Storcken (2007); Suzumura 2007; Storcken (2008); Islam, Mohajan and Moolio (2009); Miller (2009); Brams, Fishburn (1978) and Sato (2009) .

The paper is organized as follows: In section-2 we have included Social Choice and Political Relations with society in the light of Game Theory. Here we have introduced some examples and definitions and have been duly analyzed the portion in some detail. In section-3 we have included Unitary and Federal Democracy. Here we have shown the differences, advantages and disadvantages with some mathematical calculations. In section-4 we have included International Relations between two adversary countries. Here we have used game theory to explain to create a good relation among the all countries of the world and have tried to give a suggestion to create a peaceful society. In sections-5 and -6 we have described Arrow's theorem in some detail. Arrow's theorem plays an important role in Economics, Political Science and Sociology. We have tried to describe the theorem easier way so that everybody can understand its importance.

2. SOCIAL CHOICE AND POLITICAL RELATION WITH SOCIETY IN THE LIGHT OF GAME THEORY

We like to study fables and myths. Mathematical models in Social Science are like these types of fables or myths, which we read to understand the problems of the society and the precise ways to solve them by mathematical models. In this section we focus on game-theoretical models to describe Social Choice and Political Institutions. We begin with a simple story to describe Game Theory (Fudenberg and Tirole 1991).

Let us consider an island and in that island, there is a forest. In that forest there are only two kinds of animals, deer and hare. Suppose there are two hunters in that island. If both of them hunt a deer, they will share equally. If both hunt for hare they each will catch one hare. If one hunts for hare while the other for deer then the former will catch and the later will catch nothing. Let the cost of a deer is $100 and that of a hare is $10. So, each of the hunters prefers half a deer to one hare.

It is a simple example of a game theory. The hunters in the game are the players. A strategy for a player is a complete plan for hunting a deer or a hare which are the players' choice. The payoff to their choice is preying what they want to maximize. Players may learn some information in the game. One of them is cooperation by both hunting deer is equilibrium. So that equilibrium (Nash 1951) is a prediction of both players' actions such that each player's action is the best for himself given that the other player is expected to do. If each player believes the other will hunt a hare, each is better off hunting himself which is a non-cooperative outcome is also Nash equilibrium.

Game theory can be defined as the study of mathematical models of conflict and cooperation between intelligent rational decision makers (Myerson 1985). By *rational* we mean that each individual's decision-making behavior would be consistent with the maximization of subjective expected utility, if the other individuals' decisions were specified. By *intelligent* we mean that each individual understands everything about the structure of the situation that we theorists understand, including the fact that all other individual understand, and including the fact that all other individuals are intelligent and rational decision makers. The game theorist's assumption is that all the individuals are perfectly rational and intelligent. The game is described into two ways, namely: (i) the strategic form or normal form and (ii) extensive form. Here we only describe about strategic form as follows: A game in strategic form is a special case of the multistage (Myerson 1984) form in which there is only one stage and each player has only one possible information state. That is it has three elements; a set of players $N = \{1, 2, \ldots, n\}$, for each player i there is the pure-strategy space S_i and payoff function u_i that gives player i's von Neumann-Morgenstern utility $u_i(s)$ for each profile $s = (s_1, \ldots, s_n)$ of strategies. For some given player i we will use all players other than i as player i's opponent by "$-i$". First we consider the finite games where $S = \times_i S_i$ is finite. Strategic form s for finite two-player games are often depicted as payoff matrices as in figure-1, where players 1 and 2 have three pure strategies each: *U*, *M*, *D* (up, middle and down) and *L*, *M*, *R* (left, middle and right), respectively; i.e., $s_1 = \{U, M, D\}$, $s_2 = \{L, M, R\}$. The first entry in each box is player 1's payoff for the corresponding strategy profile; the second is player 2's.

	L	*M*	*R*
U	5, 4	6, 2	8, 3
M	3, 2	10, 5	4, 8
D	4, 0	11, 8	3, 10

Figure-1: Strategic Form Game.

A mixed strategy σ_i is a probability distribution over pure strategies. The space of player i's mixed strategies is denoted by $\sum_i$ where $\sigma_i(s_i)$ is the probability that σ_i assigns to s_i. The space of mixed-strategy profiles is denoted by $\sum = \times_i \sum_i$, with element σ. The players are assumed to randomize independently in a game without communication, so that player i's expected payoff to profile σ would be

$$u_i(\sigma) = \sum_{s \in S} \left(\prod_{j=1}^{n} \sigma_j(s_j) \right) u_i(s),$$

when the players used the randomized strategies $(\sigma_1, \ldots, \sigma_n)$. A combination of randomized strategies $(\sigma_1, \ldots, \sigma_n)$ is a Nash equilibrium if, for every player i and every randomized strategy σ_i', $u_i(\sigma_1, \ldots, \sigma_n) \geq u_i(\sigma_1, \ldots, \sigma_{i-1}, \sigma_i', \sigma_{i+1}, \ldots, \sigma_n)$. That is, each player i can not increase his expected payoff by using any other randomized strategy σ_i' instead of σ_i, when every other player j is using σ_j. A mixed strategy profile σ^* is a Nash equilibrium if, for all players i,

$$u_i\left(\sigma_i^*, \sigma_{-i}^*\right) \geq u_i\left(s_i, \sigma_{-i}^*\right) \quad \forall s_i \in S_i.$$

Nash equilibrium is strict (Harsanyi 1973) if each player has a unique best response to his rivals' strategies. That is, s^* is a strict equilibrium if and only if it is a Nash equilibrium and for all i and all $s_i \neq s_i^*$,

$$u_i\left(s_i^*, s_{-i}^*\right) > u_i\left(s_i, s_{-i}^*\right).$$

By definition, a strict equilibrium is necessarily a pure strategy equilibrium.

Here we observe that in figure-1 no matter how player 1 plays, *R* gives player 2 a strictly higher payoff than *M* does which is called strategy *M* is strictly dominated. Therefore a rational player 2 should not play *M*. Again, if player 1 knows that player 2 will not play *M*, then *U* is a better choice than *M* or *D*. If player 2 knows that player 1 will not play *M*, then player 2 should play *L*. This process is called iterated strict dominance.

Now we discuss varying the strategies of a single player i while holding the strategies of his opponents fixed. Let $s_{-i} \in S_{-i}$ denote a strategy selection for all players but i, and we can write

$$(s_i', s_{-i}) = (s_1, \ldots, s_{i-1}, s_i', s_{i+1}, \ldots, s_n).$$

Similarly, for mixed strategies we can write

$$(\sigma_i', \sigma_{-i}) = (\sigma_1, \ldots, \sigma_{i-1}, \sigma_i', \sigma_{i+1}, \ldots, \sigma_n).$$

Definition: Pure strategy s_i is strictly dominated for player i if there exists $\sigma_i' \in \sum_i$ such that

$$u_i(\sigma_i', s_{-i}) > u_i(s_i, s_{-i}) \quad \forall s_{-i} \in S_{-i}.$$

The strategy s_i is weakly dominated if

$$u_i(\sigma_i', s_{-i}) \geq u_i(s_i, s_{-i}) \quad \forall s_{-i} \in S_{-i}.$$

A two-player zero-sum game is a game such that $\sum_{i=1}^{2} u_i(s) = 0$ for all s; where one player wins and the other loses. This type of game studied in game theory but most of the games used in social sciences are non-zero-sum.

Now we describe a game named "The Battle of the Sexes". The Battle of the Sexes is a two player coordination game is used in the game theory. Imagine a couple. The husband would most likely to go to the cricket game. The wife would most likely to go to the film show. Both would prefer to go to the same place rather than different ones. The payoff matrix labeled in figure-2 below is a The Battle of the Sexes where the wife chooses a row and the husband choose a column. This game has three Nash equilibria as described in the following figure-2.

Now we introduce a simple game theoretic model that tells how political institutions may be founded. Let us consider a simple "Battle of Sexes" game shown in figure-2 (Myerson 1996, 2004). The two players in this game are called player A and player B, must independently choose one of the two possible strategies: to grab or to defer.

		Player B	
		grabs	defers
Player A	grabs	0, 0	8, 4
	defers	4, 8	0, 0

Figure-2: A Simple Battle of Sexes Game.

If both the players grab or defer then neither player gets anything; if exactly one player grabs then he gets payoff 8 while deferential player gets payoff 4.

This simple game has three equilibria. There is equilibrium in which player A grabs while player B defers, giving payoffs (8, 4). There is another equilibrium in which player A defers while player B grabs, giving payoffs (4, 8). There is a third equilibrium in which both players independently apply the same randomize strategy. Let us consider the probability of player A grabs is x, so the probability of deferring of them is $(1-x)$. Hence, from the figure-2 we get,

$$0.x + 8(1-x) = 4.x + 0.(1-x) \Rightarrow x = \frac{2}{3}.$$

Therefore, the probability of grabbing of each player is $\frac{2}{3}$ and the probability of deferring is $\frac{1}{3}$.

In this randomized equilibrium the payoffs are as follows:

The payoff of player $A = \frac{1}{3}\left(0\times\frac{1}{3}+4\times\frac{2}{3}\right)+\frac{2}{3}\left(8\times\frac{1}{3}+0\times\frac{2}{3}\right)=\frac{8}{9}+\frac{16}{9}=2\frac{2}{3}$, and the payoff of player $B = \frac{1}{3}\left(0\times\frac{1}{3}+8\times\frac{2}{3}\right)+\frac{2}{3}\left(4\times\frac{1}{3}+0\times\frac{2}{3}\right)=\frac{16}{9}+\frac{8}{9}=2\frac{2}{3}$. Hence in this randomize equilibrium, the expected payoff are $\left(2\frac{2}{3},\ 2\frac{2}{3}\right)$ which is worse for both players than either of the non-symmetric equilibria.

Now consider the island mentioned above with a large population of individuals. Everyday they randomly matched into pairs and play the simple Battle of Sexes once. Each player's objective is to maximize a long-run discount average of his sequence of payoffs from these daily Battle of Sexes matches. Symmetric randomized equilibrium mentioned above is a long-run equilibrium for every player. But some of the players will want to maximize their payoffs by breaking symmetry among the matched players. This will create an anarchic state. Instead of this they will share an understanding about who should grab and who should defer. At this stage they develop an understanding that each player will grab when grabbing is in favor of him and the other player do better by deferring (getting 4 better than 0) and thus they play the game without any disturbance, which is also a self enforcing equilibrium. Such an ownership creates a problem that some players fail to cover many matching situations where they have no clear concept of the game. Instead, they can create an easier method so that everybody can understand the grabbing rights. So, all the islanders again need to meet together in a public meeting and ratify new principles to resolve the problems. If they fail to establish a suitable law then they can call one of the individuals to act as a neutral arbitrator. For being neutral the arbitrator can toss a coin and may recommend that the player A should grab and player B should leave if the coin is Heads and vice-versa if the coin is Tails. In this case, the lower payoff player B could apply to the arbitrator for second trial. Therefore for the players to be coordinated by a random device, they need some way to focus themselves on one randomization that can not be repeated or appealed which will be focal arbitration (Schelling 1960).

In this situation the islanders need a leader to provide focal arbitration. The leader must be any eligible person from the islanders and can be elected by a public election. The islanders might obey his instruction as long as everyone else is expected to obey him. The leader will instruct them everyday who should grab and who should defer, which is of course a self enforcing equilibrium. The islanders have right to remove a leader in the next election when they observe that their leader is a corrupt person.

Of course, the real world is very different from the simple island of this fable. But as in this island, coordinated games with multiple equilibria are pervasive in any real society. Thus, any successful society must develop leadership structures that can coordinate people's expectations in situations of multiple equilibria. So the first point of this fable is the basic social need for leadership and political institutions can provide it to their people. The process of selecting a constitution can be viewed as an equilibrium selection problem.

Suppose that the players' payoff in our model can also be interpreted as resources that increase their long-term reproductive fitness. Then an anarchic island where these resources are wasted in the symmetric equilibrium would sustain a much smaller population than another island, wherein the players have systems of authority to coordinate them as better equilibria. If players from highly populated islands can colonize under populated islands, taking with them their cultural system of focal-equilibrium selection, then an archipelago of such islands should eventually be inhabited only by people who have systems of authority to coordinate them in matches where there are multiple equilibria. Thus, in any cultural tradition that has survived into the modern world, we should expect to find generally-accepted systems of rights and authority that provide effective focal coordination in most of the important games with multiple equilibria that may arise in daily life.

The second point of this fable is that the effectiveness of a political institution may simply be derived from a shared understanding that it is effective. The remark that our islanders might choose their leader by a general public election is meant to suggest that the rules of any higher-order social institution may itself be sustained as equilibrium in a broader and more fundamental game that has an enormous multiplicity of equilibria. Applied social theorists must understand that there are games within games in the real institution which itself can be the subject of game-theoretic analysis. Any political system may be understood as one of many possible equilibria of constitutional selection (Hardin 1989). To the extent that political leaders can develop general rules and guide lines for the creation of new social and economic institutions throughout the society, the political process of selecting a constitution may be viewed as the equilibrium-selection problem to solve all other equilibrium-selection problems in the society.

The third point of this fable is that norms of justice may be largely sustained by systems of lower-order interactions among small group of people. Although resources could be rationally expended in paying leaders or in contests for leadership, our model also admits equilibria where justice is provided at no cost in every dyadic interaction. Any deferring player will obtain payoff 4 when justice demand it, while grabbing player will obtain payoff 8. So the higher-class societies always want to grab and lower-class societies always compel to defer if laws in the society is deteriorated. Blatant thievery may indeed be the difficulty of grabbing that other people expect to be deferring for them with such a bully. So, the establishment of justice in our island is an essential issue and the political institutions must be impartial, lustful and selfishness to build up a standard society in this island. The impartial leader will educate every people to become rational and intelligent to develop a civilized society in the whole island. But in the real society leaders or political institutions are not so. Everyone including leader is selfish materialist; i.e., everybody in the society wants his own maximum social welfare.

The forth point of this fable is the concept of a boundary of the state wherein even small infractions of a boundary typically evoke costly defensive responses. Schelling (1960) showed that this theory of focal equilibria may provide a basis for understanding the logic of such rigid boundaries. If a player fails to vigorously fight against even a small violation of a perceived boundary might lead others to believe that the player would also surrender much larger areas.

Let us explain this with an example from Myerson (2004). Consider the two players in figure-2 who will match everyday many instances of our "Battle of Sexes" game, but each of these matches will be located at different places in the island. We may imagine that the payoffs in these matches are fruits that ripen each day on bushes scattered throughout the field. Suppose that the player A's home is in the east of the field, player B's home is in the west of the field, and there is an old fence that crosses the field from north to south. So the player A only expected to grab in the east of the fence, and player B is expected to grab the west of the fence. Now suppose that one bush has grown right through the fence, so there is confusion about which side it is on. Then both of the players will want to grab it simultaneously. If any player ever left the boundary without grabbing then the other player will grab it confidently always and he also will grab fruits of both sides of the fence successfully in all future matches. Hence saving boundary is an important issue and patriotic symbols for the islanders of both states. The islanders will sacrifice their best efforts to save the boundary of their own state otherwise they will not be able to save their state.

3. UNITARY AND FEDERAL DEMOCRACY OF TWO STATES AND COMPARISON BETWEEN THEM

Let us suppose the island is divided into two states, namely the east and the west (Myerson 2006). Let us consider further that at present an elected leader with unitary democracy rules the east state and an elected leader with federal democracy rules the west state. First we analyze the unitary democracy of the east state. The leader is elected by public for a fixed period and then run for re-election again in each period until rejected by the voters. In each period he may rule honorably or corruptly. Let r denote the leader's payoff each period if he rules honorably and $r+c$ be the payoff if he rules corruptly. So that c is the ruler's additional benefit in each period if he rules corruptly instead of being honorable. A politician who is out of power gets no payoff. In each period, each voter gets welfare w from the government if the leader is honorable and each voter gets 0 if the leader is corrupt. Let voter transition cost is x when they elected a new leader to oust the previous leader. So that in the case of changing a leader each voter's payoff is

$(w - x)$ if the new leader is honorable and $(0 - x)$ if the new leader is corrupt. Each politician wants to maximize the expected total discount value of payoffs, where payoffs in future periods are discounted by some discount factor ρ per period. Voters also discount their future payoffs by the same factor ρ per period. Here the parameters are $r, c, w, x, \rho > 0$ with $\rho < 1$. In equilibrium of this game, we say that democracy succeeds when the voters expect that their leaders will always serves honorably, with probability 1 and the voters in this case get maximum payoffs. So the voters always elect a leader, who is honorable and oust a leader, who is ever deviated to corruption. When the leader is expected to always serve honorably, the voters' expected discounted value of future payoff is $w(1 + \rho + \rho^2 + \ldots) = \frac{w}{1-\rho}$, and the leader's expected discounted value of future payoff is $\frac{r}{1-\rho}$. In this case for success of democracy to be an equilibrium, $\frac{w}{1-\rho} \geq x$ and $\frac{r}{1-\rho} \geq r + c$.

Hence the voters want to replace corrupt leaders, and leaders prefer a long honorable career over a short corrupt career.

If the voters always re-elected a leader with probability 1, regardless of whether acting honorably or corruptly then of course the democracy will be frustrated and the equilibrium is maximum for the leader. Again democracy will fail if the voters expect that their leader will always act corruptly and they do not want to oust the leader, since they have to expend a transition cost x, if they oust the corrupt leader. So that the corrupt leader always takes positive benefit c, if the democracy is frustrated and fails it will be called bad equilibrium. In this situation let be a small probability $p > 0$ that a politician may be intrinsically virtuous, and serve honorably. A politician who is not intrinsically virtuous may be called normal, so that any politician has probability $(1 - p)$ of being normal. For a bad equilibrium we get,

$$p \leq \frac{x(1-\rho)}{w}; \text{ i.e., } x \geq \frac{pw}{1-\rho}.$$

Therefore, transition cost x is greater than the expected gain $\frac{pw}{1-\rho}$ from better government in the unlikely p - probability event of getting a new leader who is virtuous. Hence possibility of democratic failure still exists if the probability of intrinsic virtue is small enough.

The ultimate fate of such state will be that the democracy will be abolished and the current leader will rule forever. Then the voters would get payoffs w forever if the current leader is virtuous and they get 0 forever if that leader is normal.

Now we describe the federal democracy of the west state where each of N provinces has an elected leader called a governor, and there is also an elected leader of the nation called the president (or prime minister). In federal democracy at the beginning of each period, voters in the nation first choose a president, and then voters in each province choose a governor of their province. Then elected leaders both national and provincial will serve honorably or corruptly. A virtuous leader of course serves honorably. As above a normal president gets payoff r_1 or $r_1 + c_1$ each period, depending on whether honorable or corrupt. Voters get a payoff w_1 from the national government when the president serves honorably, but they get 0 from corruption, and they pay transition cost x_1 whether they elected a new president. Similarly, a governor gets payoff r_0 or $r_0 + c_0$ each period, depending on whether honorable or corrupt. Provided voters get a payoff w_0 from their provincial government when the governor serves honorably, but get 0 from corruption, they pay a transition cost x_0 whenever they change a new governor. As before each politician has probability p of being virtuous type, and otherwise normal. Voters and politicians discount their future payoffs by the discount factor ρ in each period. We assume that the provincial governors and vice-versa do not influence national election. In federal equilibrium, the democracy in either level succeeds if the elected leader always serves honorably, with probability 1. The democracy is frustrated if they (leaders) expect to be re-elected always with probability 1, even though they serve corruptly. The democracy fails if the voters expect that the leaders that they elect will always serve corruptly if they are normal. As before we can write for provincial democracy $p < \frac{x_0(1-\rho)}{w_0} < 1$ and $r_0 + c_0 < \frac{r_0}{1-\rho}$, and similarly in national democracy $p < \frac{x_1(1-\rho)}{w_1} < 1$ and $r_1 + c_1 < \frac{r_1}{1-\rho}$.

We assume that a politician would always prefer being president over being governor, so that $r_1 > r_0 + c_0$.

We observe that there are multiple equilibria in this case. First we consider an equilibrium where provincial democracy is frustrated and fails. In this case corrupt governors would not be re-elected, so that all the governors would serve honorably. The national voters always expect that they will re-elect a corrupt leader.

Again we consider an equilibrium where provincial democracy is frustrated and fails but national democracy succeeds. In this case a corrupt president would not be re-elected so that any president would serve honorably. All the governors expect to be re-elected again, since they know that voters do not elect new leaders. If a governor serves honorably he has a little chance of winning for higher national offices, therefore, the governors always trend to be corrupted. In both equilibria there is an inconsistency between national and provincial politics.

There is a third equilibrium where democracy always succeeds at both the provincial and national levels. In this equilibrium the governors and the president always serve honorably because they know that otherwise they would not be re-elected. In a real democratic state the democracy cannot consistently be frustrated at both levels in a federal system which is the feature of a true democratic state.

From the above discussions we can compare the unitary and federal democracy as follows: In unitary democracy there is a notable chance of electing a corrupt leader, but in federal democracy it is not possible being so easily. If the citizens are rational and intelligent then there is a little chance of electing corrupt leaders. In both cases, the citizens will not re-elect the corrupt leaders. Hence we can suggest that the unitary democracy is comparatively fragile but the federal democracy is stronger and lives long.

4. INTERNATIONAL RELATION BETWEEN TWO ADVERSARY COUNTRIES

In a dangerously perturbed world we need to create peaceful atmosphere for the innocent common people. Expansions of nuclear weapons are threat to the innocent people. In 1945 two towns of Japan, Hiroshima and Nagasaki, were destroyed by atom bombs killing millions of innocent citizens. Here we use Prisoners'

Dilemma game to describe two rival countries behavior as shown in figure-3 (Schelling 1960; Myerson 2007). Two countries' problems have to be solved by the successful deterrent strategy, which is basis on balance between resolve and restraint provided that both countries have sufficient knowledge about these. As before this game has two players, player A and player B. For interpretation, let A stands for America and B for Iran. In this game, each player must simultaneously choose one of two possible actions: cooperation or aggression.

		Player B	
		Cooperation	Aggressive
Player A	Cooperation	0, 0	- 10, 2*
	Aggressive	2*, -10	-5*, - 5*

Figure-3: Prisoners' Dilemma Game.

Here asterisks (*) indicate each player's best payoff. The cell that has two asterisks is a Nash equilibrium of the game which is $\left(-5^{*},-5^{*}\right)$; i.e., both players must be aggressive is a unique Nash equilibrium of the game. Of course for both players' cooperation would be better off but mutual cooperation is not equilibrium, as each player will always be tempted to aggression. Consider that player A will move according to the motion of B. When player A gets to move second after observing what B does, player A has four possible strategies which are listed in figure-4 (Myerson 2007).

A's strategy	B cooperative	B aggressive
A is cooperative always	A cooperative	A cooperative
A does the same as B	A cooperative	A aggressive
A does the opposite of B	A aggressive	A cooperative
A is aggressive always	A aggressive	A aggressive

Figure-4: The Four Strategies of A'S Action When A Can Observe B'S Prior Action.

From figures-3 and 4 we can form figure-5 as follows:

	B cooperative	B aggressive
A is cooperative always	$0,\ 0$	$-10,\ 2^{*}$
A does the same as B	$0,\ 0^{*}$	$-5^{*},\ -5$
A does the opposite of B	$2^{*},\ -10$	$-10,\ 2^{*}$
A will always be aggressive.	$2^{*},\ -10$	$-5^{*},\ -5^{*}$

Figure-5: A Game Where Player A Moves After Observing B'S Action.

In each row of figure-5, an asterisk in the second number indicates the best payoff of B. Observe that there is only one such asterisk where A does the same as B. So the player A has one deterrent strategy that motivates B to act cooperatively. But in this case A would get highest payoff 2 when A would act aggressively. So A does not want to actually follow his deterrent strategy when B cooperates and B should not believe that A would use this deterrent strategy, unless A can somehow constrain himself to follow this strategy. Without such restraint, this game still has only an equilibrium, where both players are aggressive and both get payoff -5. For being B cooperative A must make a credible commitment and to be sure this there need some outside force such as UN to restrain player A from acting aggressively when B has cooperated. Note that player A behaves as like in figure-5 always with a different player B each time that is cooperative, but it is impossible if player B is same always.

Suppose that player A has a reputation of always being cooperative. So if player A ever lost that reputation, by acting aggressively against a cooperative player B, then the world will believe that A will betray always. Let the value of reputation be R then if A is aggressive then we will subtract reputation value R. So that by the above scenario we can form the following figure-6, where as long as the reputation value R is greater than 2, there is a good equilibrium in which B is cooperative and A does the same as B (figure-6). As before in this game has a bad equilibrium where both are aggressive always both will get the bad payoff '-5'.

	B cooperative	B aggressive
A is cooperative always	$0^*, 0$	$-10, 2^*$
A does the same as B	$0^*, 0^*$	$-5^*, -5$
A does the opposite of B	$2-R, -10$	$-10, 2^*$
A is aggressive always	$2-R, -10$	$-5^*, -5^*$

Figure-6: Player A Loses Reputation R If A Is Aggressive.

So remembering the bad equilibrium and lost of reputation both players will focus on the better equilibrium according to Schelling's focal-point effect. The world wants this type of equilibrium since it tends to a focal point say *peace*. Hence we can say that if America is always cooperative with Iran, then Iran as well as other countries will be cooperative, and the innocent people will suffer no more. If America acts aggressively with Iran and other countries of the middle-east then the whole Arab countries may be unified. In that case America may pay a greater value for this aggression.

We have mentioned that game-theoretic analysis is based on an assumption that people are rational. Of course nobody is perfectly rational. But all the people have common knowledge that conflict gives no benefit but destruction. In the history we have seen that there are people in the world who irrationally drawn to violence and destruction. For example Psychopathic militarists like Hitler become a threat to our civilization only when ordinary rational people become motivated to support them as their leaders. Now we can say that American people are rational and can elect a president who will be cooperative to the leaders of the other countries. As America is the most powerful country at this moment, the citizens of America should act rationally otherwise they may lose the reputation forever.

5. ARROW'S IMPOSSIBILITY THEOREM

Arrow's impossibility theorem (Arrow 1963) is very subtle but delicate. Arrow showed that the preferences of many individuals be aggregated into social preference but there is a flaw in this aggregation. Because a social welfare function cannot be derived by democratic vote to reflect the preferences of all the individuals in the society. Here we will try to give a very simple version of Arrow's impossibility theorem.

Before going to discuss Arrow's impossibility theorem we need to discuss some definitions which are easy but will help those who are new in this field (Arrow1963; Sen 1970; Islam 1997; Islam, Mohajan and Moolio 2009: Spring).

5.1 Preference Relations in Social Choice

In this section we consider the set of all n-tuples of real numbers which is denoted by R^n is called n-dimensional Euclidean space. A typical element or a vector in this space is denoted by $x = (x_1, x_2, \ldots, x_n)$ where $x_i\ (i = 1, 2, \ldots, n)$ are real numbers. Suppose $y = (y_1, y_2, \ldots, y_n)$ be another vector then

$x \geq y \Rightarrow x_i \geq y_i,\ \forall i$

$x > y \Rightarrow x \geq y$ but $x \neq y$; that is, x_i is different from y_i for at least one i,

$x >> y \Rightarrow x_i > y_i,\ \forall i$.

Suppose two bundles of commodities are represented by the vectors x and y. The components represent amounts of a different commodity in some unit, such as kilogram. We assume that one prefers the bundle x to the bundle y or he prefers y to x, or he is indifferent to the choice between x and y. We can write these possibilities, respectively, as follows: xPy, yPx, xIy.

Sometimes we use the notation xRy to mean that either x is preferred to y or x is indifferent to y, so that y is not preferred to x. If xPy then it is not necessary that all the commodities of x are greater than all the corresponding components of y. We can write that it is not necessary that $x >> y$ or even $x \geq y$. If x is not greater than y we write $\sim xRy$.

Let $S = \{x, y, z\}$ be a set consists of three bundles of commodities. The following are important elements in our discussion.

i) Reflexivity: $\forall\ x \in S; xRx$.

ii) Completeness: $\forall\ x, y \in S\ \&\ x \neq y \Rightarrow xRy \text{ or } yRx.$

iii) Transitivity: $\forall\ x, y, z \in S, \text{if } xRy\ \&\ yRz \Rightarrow xRz.$

iv) Anti-symmetry: $\forall\ x, y \in S, \text{if } xRy\ \&\ yRx \Rightarrow x = y.$

v) Asymmetry: $\forall\ x, y \in S, \text{such that} xRy \Rightarrow\ \sim (yRx).$

vi) Symmetry: $\forall\ x, y \in S; xRy \Rightarrow yRx.$

We also use the following notations:

For all $x, y, z \in S$;

i) If xPy then $xRy\ \&\sim yRx$.

ii) If xIy then xRy & yRx.

iii) If xPy, yRz then xPz.

iv) If xIy, yIz then xIz.

5.2 Utility Function

We now define the utility function as $u(x) = u(x_1, x_2, \ldots, x_n)$. In preference relation we can write $u(x) > u(y) \Leftrightarrow xPy$.

Let us consider a fixed vector x_0, and consider the set of all the vectors x which are preferred to x_0. If we denote this set by $V(x_0)$, we can write

$$V(x_0) = \{x : xPx_0\}.$$

For the utility function it can be written as,

$$V(x_0) = \{x : u(x) > u(x_0)\}$$

where $V(x_0)$ is a convex set.

5.3 Maximal Set and Choice Set

An element $x \in S$ is a maximal element of S with respect to R if

$$\exists\, y \in S, \sim (yPx).$$

The set of maximal elements in S is called its maximal set, and is denoted by $M(S, R)$.

An element $x \in S$ is a best element of S with respect to R if

$$\forall\, y \in S \Rightarrow xRy.$$

The set of best elements in S is called its choice set, and is denoted by $C(S, R)$.

A best element is also a maximal element but not vice-versa. Since if xRy, $\forall\, y \in S$ then $\sim yPx$.

On the other hand, if neither xRy nor yRx then x and y are both maximal elements of the set $\{x, y\}$, but neither is a best element. Hence we can write,

$$C(S, R) \subseteq M(S, R).$$

5.4 The Pareto Criterion

Let us consider, $X = \{x, y, z, \ldots\}$ be a set of alternatives. Then for any two alternatives x and y if everyone in the society satisfy xIy then the society should satisfy xIy and we shall say that the society has *Pareto-wise* indifferent between x and y. If at least one individual satisfies xPy and the other satisfy xRy then the society should satisfy xPy and we say that x is Pareto-wise better than y. An alternative x belong to the set X will be described as *Pareto-optimal* if there is no other alternative in the set which Pareto-wise better than x. Most of the modern welfare economies are based on Pareto optimality. The optimality of a system is judged in terms of whether it achieves Pareto-optimality or not. But we cannot proceed too more. Suppose one individual prefers x to y and another prefers y to x, and the rest of the individuals are x vis-à-vis y and no matter how many of the individuals are there. In this case we cannot compare the society using the Pareto rule. Sometimes an economy or society can be Pareto-optimal but may be dangerous and fully annoying. For example, some people are rolling in luxury and others are near it and as long as the starved cannot be made better off, without cutting into the pleasures of the rich people, but still the economy can be optimal.

5.5 Method of Majority Decision (MMD)

When we go from individual orderings to social preferences we call it collective choice rule (CCR). The MMD is one such CCR. The MMD means the social welfare function in which xRy iff $N(x,y) \geq N(y,x)$, where $N(x,y)$ be the number of individuals such that $xR_i y$. MMD may be transitive; i.e., if xRy, yRz then xRz. If $x = y = z$ then $xRy \Rightarrow xRx$, $\Rightarrow N(x,x) \geq N(x,x)$. In this case also MMD is transitive. But sometimes MMD is not transitive. For example, in ABC voter paradox (Here A for Arrow, B for Black and C for Condorcet; as they first drew attention to such paradox) suppose we have a community consisting of three individuals A, B and C. Assume that they have three alternatives x, y, z from which to choose. Let x, y and z stands respectively for hot war, cold war or peace with another group of individuals. If A prefers x to y, and y to z then we write

$x_A P y_A P z_A$ etc.

Here we omit indifference between two alternatives; that is, for x and y we have xPy or yPx. We assume that choices x, y and z are transitive; that is, xPy and $yPz \Rightarrow xPz$.

For voter paradox, suppose the preference relation for A, B and C are as follows;

$$x_A P y_A P z_A \quad (1a)$$

$$y_B P z_B P x_B \quad (1b)$$

$$z_C P x_C P y_C . \quad (1c)$$

Now we want to impose two conditions which are (i) the relation should be transitive and (ii) the relation should satisfy the majority rule. From (1) we see that x is preferred to y by A and C, and the majority rule x is preferred to y by the group. Again, we see that y is preferred to z by A and B, again by the majority rule y is preferred to z by the group. Since we claim that the group choice be transitive, so that x will be preferred to z by the group. If we now require that the group choice be transitive, we deduce that x is preferred to z by the group. However, from (1b-c) we see that in fact z is preferred to x by B and C, hence the majority rule z should be preferred to x. Thus we see that in the situation that the individual choice is given by (1a-c) it is not possible to impose the requirements of transitivity and majority rule simultaneously, although these conditions are fairly reasonable.

The above problem expresses the fact that certain difficulties arise when we try to work out the preference of a group from those of the individuals in it, even when one wants his reasonable requirements to be satisfied. Arrow's theorem (will be discussed below) deals with such impossibility of finding group preference.

5.6 Social Welfare Function (SWF)

The definition of SWF by Arrow is as follows:

A CCR that specifies orderings for the society is called a social welfare function *F*. SWF is such a rule that each social collective choice rules such that each social preference that is determined is an ordering. Arrow's SWF is such a particular type of CCR that each social preference satisfies an ordering (reflexive, transitive and complete). It must always work, no matter what finite set of alternatives and preference profiles; i.e., if xPy at the start it always remains the same. In SWF there must be no dictator. Let $Y = \{a_1, a_2, ..., a_n\}$ denote a finite set of alternatives or social choice options among which the voters must select one and let $R(Y)$ denote the set of strict linear rankings on Y. Let $N = \{1, 2, ..., n\}$ be a finite set of individual voters. A function $f : R^N \to Y$ will be called a social choice function. A member of R^N is called a profile of rankings and its ith component is called individual i 's ranking.

A SWF is a function $f : R^N \to R$ which aggregates voters' preferences into a single preference order on Y. The N -tuple: $(R_1, ..., R_N)$ of voters' preferences is called a preference profile.

The methods of transforming preference profiles into winners; i.e., mappings from the set of possible preference profiles into the set of alternatives is called voting procedures. For each preference profile the mapping procedures have a single winning alternative. Such a mapping is called a social decision function (SDF). The SWF first studied by Arrow, are the rules for transforming preference profile into social preference orderings or rankings. Arrow indicates that there exists no satisfactory SWF (will be discussed below). A satisfactory SDF should not be a dictatorship (Feldman 1979). Gibbard (1973) and Satterthwaite (1973, 1975) independently proved this as follows: "If a satisfactory social decision function is one which is always immune to manipulation and which is non-dictatorial, there is no satisfactory social decision function".

5.7 Social Choice Function and Monotonic Function

Let $N = \{1, 2, ..., n\}$ be the set of individual voters, and let $Y = \{x, y, z, ...\}$ be the complete and transitive finite set of alternatives. Let $L(Y)$ denote the set of strict transitive ordering of the alternatives in Y and $L(Y)^N$ denote the set of profiles of such preference orderings, one for each individual voter. A function $f : L(Y)^N \to Y$ will be called a social choice function. A social choice function f is monotonic if whenever $f(L_1, ..., L_N) = x$ for any alternative x and for every individual i, and every alternative y the ranking L'_i ranks x above y if L_i does, then $f(L'_1, ..., L'_N) = x$.

5.8 Prerequisites of Arrow's Impossibility Theorem

For simplicity let us consider there are two individuals in the society and three social alternatives x, y, z. For the preference orderings for individual 1 or 2 there are exactly $6 \times 6 = 36$ different constellations of individual preferences possible in the society (figure-7) where alternatives are ordered from top to bottom (Feldman 1974).

	1 2	1 2	1 2	1 2	1 2	1 2
1st	x x	x x	x y	x y	x z	x z
2nd	y y	y z	y z	y x	y x	y y
3rd	z z	z y	z x	z z	z y	z x
1st	x x	x x	x y	x y	x z	x z
2nd	z y	z z	z z	z x	z x	z y
3rd	y z	y y	y x	y z	y y	y x
1st	y x	y x	y y	y y	y z	y z
2nd	x y	x z	x z	x x	x x	x y
3rd	z z	z y	z x	z z	z z	z x
1st	y x	y x	y y	y y	y z	y z
2nd	z y	z z	z z	z x	z x	z y
3rd	x z	x y	x x	x z	x y	x x
1st	z x	z x	z y	z y	z z	z z
2nd	x y	x z	x z	x x	x x	x y
3rd	y z	y y	y x	y z	y y	y x
1st	z x	z x	z y	z y	z z	z z
2nd	y y	y z	y z	y x	y x	y y
3rd	x z	x y	x x	x z	x y	x x

Figure-7: The preference orderings for individual 1 or 2 there are exactly 36 different constellations of individual preferences possible in the society.

Before going to discuss Arrow's impossibility theorem we have to study the following requirements.

i) Completeness and transitivity: We have defined these conditions in section-3 as: for any pair of alternatives x and y then either xRy or yRx must hold, and for any triple x, y, z we have for xRy and yRz must imply xRz. The social preference relations generated by a CCR must be complete and transitive. The requirement says that a CCR must always permit social choices between alternatives, and that social choices must be consistent, or not inherently self-contradictory. Several well known collective choice rules do not generate complete and transitive social preference relations. For example, unanimous voting produces incomplete social rankings: if individual 1 prefers x to y and 2 prefers y to x neither alternative wins a unanimous vote over the other. Majority voting produces non-transitive social rankings (See definition MMD).

ii) Universality or Unrestricted Domain (U): In social choice theory U is a property of SWF's in which preferences of all the voters are factored into the final ordering of societal choices. Hence U is a common requirement for all social choice functions. U indicates that the SWF accounts for all preferences among all voters to yield a unique and complete ranking of social choices. The SWF must be wide enough in scope to work from any logical set of individual orderings. The Pareto principal gives a perfectly fine social orderings if the preferences of individuals are unanimous. But in the case of incomplete preference relations it will not have a social orderings thus it fails to satisfy the requirement of Arrow. Similarly, the MMD may yield intransitives in some cases so the MMD also fails for U. According to this restriction in figure-7, the rule should give us a social preference ordering for every cell not just for the easy ones, like those where there is unanimous agreement; i.e., the diagonal cells in figure-7.

iii) Pareto consistency (efficiency) or unanimity (P): The SWF must satisfy the Pareto principle in the weak form; i.e., if everyone prefers x to y, then the society must also prefers x to y. Mathematically, for any $x, y \in X$,

$$\forall\, i : xP_i y \Rightarrow xPy$$

i.e., if alternative x is ranked above y for all orderings $(R_1, \ldots, R_N)$ then x is ranked higher than y by $F(R_1, \ldots, R_N)$. Pareto consistency is a very mild requirement for a CCR. If the societies are ruled by external forces then one can not expect it to hold in societies. For example, (Feldman 1974) everyone prefers lust and gambling, on the one hand, to chastity and frugality on the other, but where, according to a Holy Book, the society state of chastity and frugality is preferable to the society state of lust and gambling. But the external forces, considering their economy, naturally would recommend lust and gambling.

iv) Independent of Irrelevant Alternatives (IIA): IIA means that the social ranking of x vis-à-vis y must depend only on individual rankings of x vs. z or y vs. w or z vs. w or any other such irrelevancy. Let Y be a set of alternatives, R be the set of social ordering and $C(Y, R)$ be the choice function, then for a single individual the choice made from any fixed environment Y should be independent of the very existence of alternatives outside of Y. Again consider, if R and R' be the relations determined by f corresponding respectively to two sets of individual preferences, $(R_1, \ldots, R_n)$ and $(R'_1, \ldots, R'_n)$. If $x, y \in Y, xR_i y \Leftrightarrow xR'_i y \;\; \forall\, i$, then $C(Y, R)$ and $C(Y, R')$ are the same. Condition IIA is the most subtle of all the requirements. Suppose society chooses democracy (d) over communism (c) and fascism (f) is their third alternative. At one stage everyone of the society suddenly changes the desirability of f but no one changes his mind about d vs. c. The independence requirement says that, if society is faced with the choice between d and c, and only those two, it must still choose d over c.

The standard example of a CCR that violates independence is the rule of weighted voting. Let the society be made up of two individuals 1 and 2. Suppose 1's initial preferences are fP_1dP_1c, while 2's initial preferences are fP_2dP_2c. Suppose a person's first choice gets a weight of 10 points, a second choice gets 7 points, and a third choice get 3 points. If the social choice is between c and d, d gets $7+7=14$, and c gets $10+3=13$ points; so d is socially preferred to c. Now let 1 become totally disillusioned with f; his ordering changes to $dP_1cP_1 f$. Now if d vs. c votes are repeated, d gets, $10+7=17$ points, and c gets, $10+7=17$ points. Society has become indifferent between d and c; even though neither 1 nor 2 changed his mind about these two alternatives.

v). Non-Dictatorship (D): It is required that the SWF should not be dictatorial. That is, there should be no such individual that whenever prefers x to y, society must prefer x to y, irrespective of the preferences else. This is called the condition of non-dictatorship. Mathematically, there is no such individual i that for every element in the domain of rule f, $\forall\, x, y \in X$ such that $xP_i y \Rightarrow xPy$. Anonymous voting systems with at least two voters satisfy the non-dictatorship property. The dictatorship is undesirable in the society. First, it is undesirable because one's worst enemy might be dictator. Second, it is not a CCR. So that dictatorship may cause the violation of human rights.

The Pareto consistency requirement says a CCR must respect unanimous opinion: if both 1 and 2 prefer one alternative to another, then society must also prefer the one to the other. For example, given the configuration of individual preferences of the 1st row, 3rd column cell of figure-7, the Pareto requirement says x and y must be socially preferred to z . Application of Pareto consistency over the entirety of figure-7 gives rise to figure-8. Each cell of this figure is produced by applying Pareto consistency to the corresponding cell of figure-7 so that any rule for generating social preferences must be entirely consistent with figure-8 (Feldman 1974).

xPy xPz yPz	xPy xPz	xPz yPz	yPz	xPy	
xPy xPz	xPy xPz zPy	xPz		xPy zPy	zPy
xPz yPz	xPz	xPz yPx yPz	yPx yPz		yPx
yPz		yPx yPz	yPz yPx zPx	zPx	yPx zPx
xPy	xPy zPy		zPx	xPy zPx zPy	zPx zPy
	zPy	yPx	yPx zPx	zPx zPy	yPx zPx zPy

Figure-8: Preference Relations Depending On Pareto Consistency.

Now consider the independence requirement. Suppose that, when applied to the constellation of preferences,

	1	2
1st	x	y
2nd	y	x
3rd	z	z

a collective choice rule gives x is socially preferred to y . Then xP_1y providing that xP_1y and yP_2x , no matter how 1 and 2 changes their feelings about the irrelevant alternative z .

Similarly, we must have yPx or xIy whenever xP_1y and yP_2x .

Now independence requires that all the cells in figure-7 where xP_1y and yP_2x must yield identical social rankings of x and y . Similarly, all the cells where yP_1x and xP_2y must yield identical social rankings of x and y . There is no presumption, however, that social x - y ranking on the yP_1x and xP_2y cells. Such *neutrality* condition is unnecessary for the proof of the theorem, although it is intuitively appealing and useful in other contents (Feldman 1974). We are now in a position to discuss Arrow's Impossibility Theorem:

6. ARROW'S THEOREM

Suppose that the set of alternatives Y has at least three elements and the conditions (*i*), (*ii*), (*iii*) and (*iv*) are satisfied. Then there exists an individual $u_k \in U$, such that

$$W(w_1, w_2, \ldots, w_n) = w_k \text{, some } k\text{, } 1 \leq k \leq n;$$

that is, the group preference coincides with that of some one (single) individual.

Brief Discussion: Here the discussion is simple and direct so that who are new in this field can easily capture the concept of the theorem. We will give very simple version of the theorem in five steps following Reny (2000) and Geanokoplos (2005). Here individuals' choices are inputs and social order is their output.

Step 1: Let us consider any two distinct alternatives $x, y \in Y$ and a profile of rankings in which x is ranked at the highest position and y is at the lowest position for every individual $i \in N$.
By Pareto efficiency x is strictly at the top of the social order (profile-1).

$$\begin{array}{cccccccccc} R_1 & \ldots & R_{n-1} & R_n & R_{n+1} & \ldots & R_N & & \underline{\text{Social order}} \\ x & \ldots & x & x & x & \ldots & x & & x \\ . & & . & . & . & & . & & . \\ . & & . & . & . & & . & \Rightarrow & . \\ . & & . & . & . & & . & & . \\ y & \ldots & y & y & y & \ldots & y & & y. \end{array}$$

Profile-1

Now change all the individual i 's rankings by keeping y above x for $i < n$, and keeping x at the very top and keeping y at the 2nd row for $i = n$ and keeping unchanged for $i > n$ in profile-1 to obtain profile-2. Here the individual n is extremely pivotal; i.e., the individual n be such that whatever alternative be at the very top of the ranking, the social choice will be that alternative. The IIA implies that in this case social order will be such that x will be at the very top of the ranking.

$$\begin{array}{cccccccccc} R_1 & \ldots & R_{n-1} & R_n & R_{n+1} & \ldots & R_N & & \text{Social order} \\ y & \ldots & y & x & x & \ldots & x & & x \\ x & \ldots & x & y & . & & . & & . \\ . & & . & . & . & & . & \Rightarrow & . \end{array}$$

$$\begin{array}{ccccccccc}
. & & . & . & . & & . & & y \\
. & & . & . & . & & . & & . \\
. & & . & . & y & \dots & y & & .
\end{array}$$

Profile-2

Now change all the individual i's rankings by keeping y above x for $i \le n$ and keeping unchanged for $i > n$ in profile-1 to obtain profile-3. The IIA implies that in this case social order will be such that y will be at the very top of the ranking.

$$\begin{array}{ccccccccc}
R_1 & \dots & R_{n-1} & R_n & R_{n+1} & \dots & R_N & & \text{Social order} \\
y & \dots & y & y & x & \dots & x & & y \\
x & \dots & x & x & . & & . & & x \\
. & & . & . & . & & . & \Rightarrow & . \\
. & & . & . & . & & . & & . \\
. & & . & . & y & \dots & y & & .
\end{array}$$

Profile-3

Step 2: Now we derive profile-2a from profile-2 by moving alternative x to the bottom of individual i's rankings and keeping y above x for $i < n$ and moving x to the second last row in i's ranking for $i > n$ but not changing the ranking of individual n. By IIA the social ranking remain unchanged in this new profile.

$$\begin{array}{ccccccccc}
R_1 & \dots & R_{n-1} & R_n & R_{n+1} & \dots & R_N & & \text{Social order} \\
y & \dots & y & x & . & & . & & x \\
. & & . & y & . & & . & \Rightarrow & y \\
. & & . & . & . & & . & & . \\
. & & . & . & x & \dots & x & & . \\
x & \dots & x & . & y & \dots & y & & .
\end{array}$$

Profile-2a

Now we derive profile-3a from profile-3 by moving alternative x to the bottom of individual i's rankings and keeping y above x for $i < n$ and moving x to the second last row in i's ranking for $i > n$ but not changing the ranking of R_n. By IIA the social ranking remain unchanged in this new profile.

$$\begin{array}{ccccccccc}
R_1 & \dots & R_{n-1} & R_n & R_{n+1} & \dots & R_N & & \text{Social order}
\end{array}$$

$$\begin{array}{cccccccccc}
y & \dots & y & y & . & & . & & y \\
. & & . & x & . & & . & \Rightarrow & x \\
. & & . & . & x & \dots & x & & . \\
x & \dots & x & . & y & \dots & y & & .
\end{array}$$

Profile-3a

Here we observed that profiles-2a and -3a differ only in individual n's ranking of alternatives x and y. Since social order at the very top is y in profile-3a, hence by the IIA social order at the very top will be either x or y. However, y is at the very top in profile-2a, then by IIA the social order at the very top must be y, a contradiction. Hence the social order at the very top in profile-2a is x.

Step 3: Let $z \in A$ be a distinct element from x and y. Now we form a new profile-4 from profile-2a where z is new alternative such that x is at the top, z is in 2nd row and y is in 3rd row in the n's ranking. Also z is in 3rd last row, y is in 2nd last row, x is in last row for $i < n$ and also z is in 3rd last row, x is in 2nd last row, y is in last row for $i > n$. By IIA social order at the very top in profile-4 is x.

$$\begin{array}{cccccccccc}
R_1 & \dots & R_{n-1} & R_n & R_{n+1} & \dots & R_N & & \text{Social order} \\
. & & . & x & . & & . & & x \\
. & & . & z & . & & . & & . \\
. & & . & y & . & & . & \Rightarrow & . \\
z & \dots & z & . & z & \dots & z & & . \\
y & \dots & y & . & x & \dots & x & & . \\
x & \dots & x & . & y & \dots & y & & .
\end{array}$$

Profile-4

Step 4: Now interchange the ranking of alternatives x and y in profile-4 for $i > n$ and keep the alternatives unchanged for individuals $i \le n$ to obtain profile-5. Social ranking at the very top will be x in profile-4, hence by the IIA the social order at the very top be either x or y in profile-5. However, in social order y cannot be at the top in profile-5, since alternative z is ranked above y in every individual's ranking, and Pareto efficiency would then imply that the social order y would remain above z. Hence, x is socially top-ranked and z is socially ranked above y in profile-5.

$$\begin{array}{cccccccccc}
R_1 & \dots & R_{n-1} & R_n & R_{n+1} & \dots & R_N & & \text{Social order} \\
. & & . & x & . & & . & & x \\
. & & . & z & . & & . & & . \\
. & & . & y & . & & . & & . \\
. & & . & . & . & & . & \Rightarrow & z \\
. & & . & . & . & & . & & . \\
z & \dots & z & . & z & \dots & z & & y \\
y & \dots & y & . & y & \dots & y & & .
\end{array}$$

x … x . x … x .

Profile-5

Step 5: We observed that an arbitrary profile of rankings with x at the top of individual n's ranking can be obtained have the profile-5 without reducing the ranking of x versus any other alternative in any individual's ranking. Hence, IIA implies that the social choice must be x whenever x is at the top of individual's ranking. So, we may say that individual n is a dictator for alternative x. Since x is arbitrary; we have shown that for each alternative $x \in Y$, there is a dictator for x. But clearly there cannot be distinct dictators for distinct alternatives. Hence there is a single dictator for all the alternatives.

7. GIBBARD'S THEOREM

Allan Gibbard proved his theorem in his original seminal paper, Gibbard (1977), with several lemmas. We discuss his theorem here in easier way as per as possible following Mclennan (2008).

7.1 Discussion

Let $Y = \{x, y, z, \ldots\}$ be the complete, transitive and asymmetric ordering finite set of social alternatives and let $N = \{1, 2, 3, \ldots, n\}$ be the set of individuals or voters. The utility function is defined by $U : Y \to R$. For all $x, y \in Y$, $U(x) > U(y)$ if and only if xPy. For any finite set A, $\Delta(A)$ denotes the space of probability measures on A. Each voter wants to maximize an expected utility function,

$EU = \sum_{x \in Y} U(x)\, p(x),$ whenever $p \in \Delta(Y)$ and p is the probability of winning.

Let a preference profile is defined by $\mathbf{P} = (P_1, P_2, \ldots, P_n)$. Let $P(Y)^N$ the set of profiles. For each $i = 1, 2, \ldots, n$ let $P_{-i}(Y)^N$ be the set of $(n-1)$-tuples of preferences $(P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_n)$, thought of as configurations of preferences of the voters other than i. So that if $\mathbf{P} \in P(Y)^N$ is given $\mathbf{P}_{-i}(Y)$ will denote the $(n-1)$-tuples obtained by dropping P_i. Let $\mathbf{P}_{-i}(Y) \in P_{-i}(Y)^N$ then P_i' are given by

$$(\mathbf{P}_{-i}, P_i') = (P_1, \ldots, P_{i-1}, P_i', P_{i+1}, \ldots, P_n)$$

and is the profile obtained by combining these objects. If $\mathbf{P} \in P(Y)^N$ and P_i' are given by,

$$\mathbf{P} / iP'i = (\mathbf{P}_{-i}, P_i')$$

and is the profile obtained from $\mathbf{P}$ by replacing P_i with P_i'. Now we defined the decision scheme as

$$d : P(Y)^N \to \Delta(Y).$$

We denote the probability assigned to alternative x by the decision scheme at profile $\mathbf{P}$ by $d(x, \mathbf{P})$. Let $d = (d_1, \ldots, d_m)$ and if there are positive scalars $\alpha_1, \ldots, \alpha_m$ such that $\alpha_1 + \ldots + \alpha_m = 1$ then

$d(x, \mathbf{P}) = \alpha_1 d_1(x, \mathbf{P}) + \ldots + \alpha_m d_m(x, \mathbf{P})$ for all alternatives x and profiles $\mathbf{P}$. The decision scheme d is potentially manipulable by i at a profile $\mathbf{P}$ if $U(d(\mathbf{P} / iP'i)) > U(d(\mathbf{P}))$. If d is not manipulable then it is said to be strategy-proof. A lottery ρ is Pareto optimal *ex post* for profile C if

$\rho(x) = 0$ for alternative x. The decision scheme d is *Pareto optimific ex post* if, for every profile $\mathbf{P}$, $d(\mathbf{P})$ is Pareto optimal ex post for $\mathbf{P}$.

For a preference $\mathbf{P}$, $f(\mathbf{P})$ be the top ranked alternative. A decision scheme d is dictatorial if there exists a vector i such that $d(f(\mathbf{P}_i), \mathbf{P}) = 1$ for all $\mathbf{P} \in P(Y)^N$. A random dictatorship is a probability mixture of dictatorship. Now we can introduce the Gibbard's theorem as follows:

7.2 Theorem (Gibbard 1977)

If there are three or more alternatives and the decision scheme d is strategy-proof and Pareto optimific *ex post*, then it is a random dictatorship.

8. CONCLUDING REMARKS

This paper analyses Political Economics and Social Welfare in some details. In this paper it has been indicated how a political institution is formed, and how this serves the society by creating efficient and democratic leaders. The Nash equilibrium has been clarified by game theory, which describes and explains an essential part of the society. Game theory plays an important role also to explain more clearly the problems of Economics and Political Science. In this paper unitary and federal democracy has been discussed briefly to show their importance in the society. It is hoped that easier discussion of Arrow's theorem and Gibbard's theorem will give a better idea to the readers about dictatorship. Interested readers are requested to see Islam, Mohajan and Moolio (2009) to know about combinatorial approach and geometrical approach to Arrow's theorem. The paper is a review of other's works, but throughout the paper social matters have been discussed with simple mathematical calculations and introducing definitions wherever it was necessary.

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